# New Robust and Flexible Parameter Estimation Method

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## I. Introduction

LEAST-SQUARES estimator (LSE) and maximum likelihood estimator (MLE) are widely used in many areas, especially in estimation of aircraft stability and control derivatives.[1] However, the measurements obtained from an experiment may contain some unpredictable and undetectable readings, called "outliers," due to some uncontrollable reasons. These outliers would cause unsatisfactory estimation. Hence, a more robust estimator would be desirable for such situations.

The Huber estimator, whose performance index is based on the Huber distribution, is known to be robust.[2] The Huber distribution, however, is not very flexible and cannot cover all of the situations involving the presence of large outliers.[3] A more flexible distribution for deriving a robust estimator may be required.

The generalized $M$-estimator (GME) is derived in this Note by using a very flexible distribution. Its properties are determined from the basic theory of the $M$-estimator, and its robust recursive algorithm is also developed. In a simulation study, we have obtained good estimation results, especially in measurements with large outliers.

## II. Definition of the Generalized Distribution

The generalized beta type-2 distribution is a very flexible distribution on $[0,\infty)$.[4] For practical use, the generalized distribution on $(-\infty,\infty)$ is defined as

$$f(v|c) = \frac{a}{2bB(p,q)}\left[1+\left(\frac{|v|}{b}\right)^a\right]^{-(p+q)} \tag{1}$$

where $ap = 1$; $a \geq 1$; $-\infty < v < \infty$; $c = \{a,b,p,q\}$; $a$, $b$, $p$, and $q$ are four positive tuning constants; and $B(p,q)$ is the beta function. Table 1 shows the relationships between $f(.)$ and some known distributions. It also displays the flexibility of $f(v|c)$, which may include the Huber distribution.

## III. Generalized $M$-Estimator

Consider a location model:

$$x_i = \mu_0 + \epsilon_i, \qquad i = 1,2,\ldots,n \tag{2}$$

where $\mu_0$ is a location parameter and $x_i$ and $\epsilon_i$ the measurement and noise, respectively. Assume that $\epsilon_i$, $i = 1,2,\ldots,n$ are independent random variables with identical distribution $f_0(.)$.

An estimator $\hat{\mu}$ is called a $M$-estimator if $\hat{\mu}$ is solved by[2]

$$\frac{1}{n}\sum_{i=1}^{n}\psi(x_i - \hat{\mu}) = 0 \tag{3}$$

Table 1 Relationships between some distributions and the generalized distribution

| $a$ | $b$ | $p$ | $q$ | Distribution | Remark |
|---|---|---|---|---|---|
| 2 | $\nu^{1/2}$ | 1/2 | $\nu/2$ | $t$ distribution | $\nu>0$ |
| 2 | $\sqrt{2}q^{1/a}$ | 1/2 | $\infty$ | Standard Gauss | |
| 1 | $q^{1/a}\beta$ | 1 | $\infty$ | Double exponential | $\beta>0$ |
| 2 | 1 | 1/2 | $m-1/2$ | Cauchy | $m\geq 1$ |
| $1/m$ | $q^{1/a}$ | $m$ | $\infty$ | Light tailed | $0\leq m\leq 1$ |

where $\psi$ (.) is an arbitrary function.

In general, a bounded $\psi(.)$ would lead to robustness. A bounded $\psi$ (.) derived from $f(v|c)$ is needed for robustness and flexibility.

If we take the derivative of $f$ (.) with respect to $v$ and divide $f$ (.), then, based on the concept of the MLE, we define

$$\psi(v) = \frac{v|v|^{a-2}}{1+\left(\frac{|v|}{b}\right)^a} \tag{4}$$

where $2 \leq a < \infty$ and $0 < b < \infty$. The $M$-estimator obtained by solving Eq. (3) with $\psi$ given in Eq. (4) is called the generalized $M$-estimator.

*Lemma 1:* $\psi(v)$, as defined in Eq. (4), is an odd and bounded function.

Lemma 1 implies the robustness of the GME. Consistency is the other desirable property for a good estimator,[5] and the following theorem of consistency is given.

*Theorem 1* (Consistency): Assume the true density function $f_0(x|\mu_0)$ to be symmetric with respect to $\mu_0$ and unimodal and that $\psi(v)$ is given by Eq. (4), then

$$\hat{\mu} \to \mu_0 \qquad \text{in probability, almost surely as } n\to\infty$$

where $\hat{\mu}$ is the GME of location parameter when $n$ is fixed.

## IV. Recursive Algorithm in Dynamic System

Consider the ARX model of a dynamic system[6]:

$$A(z^{-1})y(t) = B(z^{-1})u(t) + e(t), \qquad t = 1,2,\ldots,n \tag{5}$$

where $y(t)$, $u(t)$, and $e(t)$ are the output, input, and the measurement noise, respectively. $A(z^{-1})$ and $B(z^{-1})$ are given by

$$A(z^{-1}) = 1 + a_1z^{-1} + a_2z^{-2} + \cdots + a_mz^{-m} \qquad m \geq 0$$

$$B(z^{-1}) = b_0 + b_1z^{-1} + b_2z^{-2} + \cdots + b_rz^{-r} \qquad r \geq 0$$

where $z^{-1}$ denotes the shift operator. Assume that $e(t)$, $t = 1,2,\ldots,n$ are independent and identically distributed random variables. Then the linear regression is

$$y(t) = \phi^t(t)\theta_0 + e(t) \tag{6}$$

where

$$\phi^t(t) = [u(t)\ u(t-1)\ \cdots\ u(t-r)\ -y(t-1)\ \cdots\ -y(t-m)]$$

$$\theta_0 = [b_0\ b_1\ b_2\ \cdots\ b_r\ a_1\ a_2\ \cdots\ a_m]^t$$

and where $[.]^t$ is the transpose of a matrix. Thus, the GME of system parameters vector $\hat{\theta}$ is solved by

$$\frac{1}{n}\sum_{t=1}^{n}\phi(t)\psi\left[y(t)-\phi^t(t)\hat{\theta}\right] = 0 \tag{7}$$

where $\psi$ (.) was defined in Eq. (4).

We now associate the $M$-estimator with the generalized least-squares (GLS) estimator to form the GME recursive algorithm.

Received July 3, 1991; revision received Dec. 18, 1991; accepted for publication May 26, 1992. 

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**Table 2 Estimated system parameters of recursive estimators**

| Noise | Estimator | $\hat{b}_1$ | $\hat{b}_2$ | $\hat{a}_1$ | $\hat{a}_2$ | $\hat{a}_3$ | Error, %[a] |
|---|---|---|---|---|---|---|---|
| $f_1$ | GME | 0.394 | 0.151 | −2.953 | 2.944 | −0.992 | 6.34 |
| | LSE | 0.394 | 0.163 | −2.92 | 2.881 | −0.961 | 4.45 |
| | Huber | 0.394 | 0.14 | −2.976 | 2.991 | −1.015 | 7.7 |
| $f_2$ | GME | 0.369 | 0.025 | −3.032 | 3.132 | −1.099 | 12.4 |
| | LSE | 0.369 | 0.006 | −3.081 | 3.248 | −1.166 | 15.9 |
| | Huber | 0.369 | 0.054 | −2.957 | 2.972 | −1.012 | 7.56 |
| $f_3$ | GME | 0.395 | 0.158 | −3.131 | 3.302 | −1.175 | 17.1 |
| | LSE | 0.395 | −0.434 | −4.627 | 6.308 | −2.695 | 108 |
| | Huber | 0.395 | 0.109 | −3.258 | 3.537 | −1.285 | 24.2 |
| $f_4$ | GME | 0.394 | 0.206 | −2.951 | 2.94 | −0.99 | 6.33 |
| | LSE | 0.394 | −5.399 | −17.16 | 35.32 | −19.26 | 993 |
| | Huber | 0.394 | 0.134 | −3.135 | 3.3 | −1.166 | 17.0 |

[a]Error norm $= \|\hat{\theta}-\theta_0\|/\|\theta_0\| \times 100\%$.

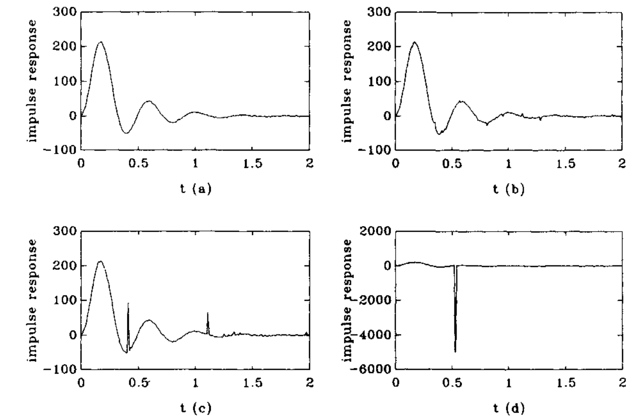


**Fig. 1 System responses under pulse input with measurement noises a) $f_1$, b) $f_2$, c) $f_3$, and d) $f_4$.**

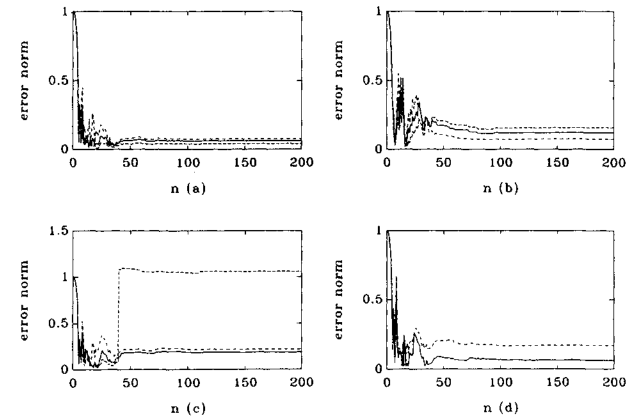


**Fig. 2 Error norm of the generalized $M$-estimator (—), least-squares (--), and Huber estimator (-··-) under measurements noises a) $f_1$, b) $f_2$, c) $f_3$, and d) $f_4$.**

Define the weighting function of the GLS based on Eq. (4) as

$$W(v) = \frac{|v|^{a-2}}{1+\left(\dfrac{|v|}{b}\right)^a} \tag{8}$$

where $2 \le a < \infty$ and $0 < b < \infty$. Thus, the GME recursive algorithm at current time $t$ is[7]

$$\hat{\theta}(t) = \hat{\theta}(t-1) + W(v)P(t)\phi(t)\left[y(t) - \phi^t(t)\hat{\theta}(t-1)\right] \tag{9}$$

where

$$P(t) = P(t-1) - \frac{W(v)P(t-1)\phi(t)\phi^t(t)P(t-1)}{1+W(v)\phi^t(t)P(t-1)\phi(t)}$$

and $v = y(t) - \phi^t(t)\hat{\theta}(t-1)$.

From the influence curve,[8] the index of robustness of an estimator, the weighting of an outlier should be reduced for robustness.

*Lemma 2:* $W(v)$ is defined as Eq. (8); then

1) $W(v)$ is a positive and even function.

2) $W(v)$ is monotone, increasing over $0 \le v \le [(a-2)/2]^{1/a}b$, and is decreasing when $v > [(a-2)/2]^{1/a}b$.

The robust recursive algorithm is obtained from Lemma 2 since the weightings of the normalized measurements in $|v| > [(a-2)/2]^{1/a}b$, which are frequently regarded as outliers, are reduced.

## V. Example

An example is given here to show the performances of the proposed recursive algorithm. Noises are generated from four density functions called the standard Gauss $f_1$, the contaminated normal $f_2$, the slash $f_3$, and the Cauchy distributions $f_4$, respectively.

Consider the ARX model of a dynamic system with $m = 3$, $r = 2$, $b_1 = 0.3933$, $b_2 = 0.1535$, $a_1 = -2.8529$, $a_2 = 2.7351$, and $a_3 = -0.8807$.

In simulation, a pulse of height 10.0 and duration 0.01 s is applied. The responses with different noises are shown in Fig. 1. The estimated parameters under $a = 2$ and $b = 9.5$ are given in Table 2.

Table 2 shows that the GME and the Huber are superior to the LSE. Moreover, the GME is better than the Huber in $f_1$, $f_3$, and $f_4$ situations. The robustness of the GME recursive algorithm is easily proved. The convergence of the proposed recursive algorithm is also shown in Fig. 2.

## VI. Conclusions

A more robust and flexible estimator of the location parameter, called the generalized $M$-estimator, is presented in this Note. Its recursive algorithm for on-line estimation in a dynamic system is also developed. Based on the foregoing discussion, such a new estimator is useful and worthy of studying.

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# Active Vibration Control of Flexible Structures with Acceleration Feedback

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## Introduction

THE vibration suppression problem is an important aspect of the overall control problem of flexible structures such as rotary and fixed-wing aircraft, as well as spacecraft. The amplitude and duration of vibration must be controlled so that the structures do not experience performance degradation or structural damage. One of the techniques that can be used for the control of flexible structures is collocated control, in which sensors and actuators are placed in the same location. Collocated control appears to be suitable for structures with distributed sensors and actuators.

Collocated velocity feedback has been studied in the past[1,2] in conjunction with the control of large flexible space structures. Collocated velocity feedback is unconditionally stable in the absence of actuator dynamics. However, in the presence of actuator dynamics, the stability condition is dependent on structural damping and frequency.[3] The combined structure and controller system will be stable if the actuator dynamics are sufficiently fast. Unfortunately, flexible structures have infinite bandwidth, whereas all actuators have finite bandwidth. Accordingly, the system stability is not guaranteed and instability could result.[3,4] The inherent lack of overall system stability renders the velocity feedback scheme less attractive.

Collocated positive position feedback overcomes the shortcomings of velocity feedback.[3] Although it is not unconditionally stable, the stability condition of positive position feedback is independent of uncertain structural damping and the actuator dynamics. Fanson and Caughey[4] conducted analytical and experimental work using a cantilever beam as an example. A large increase in damping was achieved, demonstrating the feasibility of using position feedback as a strategy to control the vibration of flexible structures. However, in positive position feedback, the stability condition may be stringent in that the magnitude of the controller gain is limited by the lowest frequencies of the structure. This may place a limit on the system damping and the overall system performance that can be achieved through positive position feedback control.

Presented as Paper 91-1015 at the AIAA/ASME/AHS/ASC 32nd Structures, Structural Dynamics, and Materials Conference, Baltimore, MD, April 8–10, 1991; received July 15, 1991; revision received Jan. 31, 1992; accepted for publication May 28, 1992. 

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Alternately, we investigate in this Note collocated acceleration feedback including finite actuator dynamics. In the next section, the stability of acceleration feedback control for flexible structures is investigated. It will be shown that when the acceleration is fed back in the collocated control, the combined structure-controller system is unconditionally stable and independent of the damping and natural frequencies of the structure to be controlled. Subsequently, vibration control of a cantilever beam is used to demonstrate the effectiveness of the control scheme.

A more general case of the control law considered in this paper has been investigated in Refs. 5–7 that covers both analytical and experimental aspects.

## Acceleration Feedback Control

An implementation of collocated acceleration feedback control for flexible structures can be written in matrix form as

Structure:

$$\ddot{\xi} + D\dot{\xi} + \Omega\xi = -P^T G\eta \tag{1}$$

Controller:

$$\ddot{\eta} + D_a\dot{\eta} + \Omega_a\eta = \Omega_a P\ddot{\xi} \tag{2}$$

where the vectors and matrices are defined as follows:

$\xi \equiv$ structure modal vector of length $n_m$

$\eta \equiv$ controller state vector of length $n_a$

$G \equiv$ gain matrix:diag $n_a \times n_a$, $g_i > 0$ for all $i$

$$= \begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_{n_a} \end{bmatrix}$$

$P \equiv$ participation matrix:$n_a \times n_m$

$\Omega \equiv$ structural frequency matrix:diag $n_m \times n_m$

$$= \begin{bmatrix} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \omega_{n_m}^2 \end{bmatrix}$$

$\Omega_a \equiv$ controller frequency matrix:diag $n_a \times n_a$

$$= \begin{bmatrix} \omega_{a_1}^2 & & 0 \\ & \ddots & \\ 0 & & \omega_{a_{n_a}}^2 \end{bmatrix}$$

$D \equiv$ structural damping matrix:diag $n_m \times n_m$

$$= \begin{bmatrix} 2\zeta_1\omega_1 & & 0 \\ & \ddots & \\ 0 & & 2\zeta_{n_m}\omega_{n_m} \end{bmatrix}$$

$D_a \equiv$ controller damping matrix:diag $n_a \times n_a$

$$= \begin{bmatrix} 2\zeta_{a_1}\omega_{a_1} & & 0 \\ & \ddots & \\ 0 & & 2\zeta_{a_{n_a}}\omega_{a_{n_a}} \end{bmatrix}$$

Then, we can prove the following theorem for the stability condition of the system.

*Theorem*: The combined structure and controller dynamics of Eqs. (1) and (2) are unconditionally Lyapunov asymptotically stable (LAS), regardless of the damping and natural frequencies of the structure.

*Proof*: By differentiating Eq. (1) with respect to time and introducing a new variable

$$\xi_n \equiv \dot{\xi} \tag{3}$$

Eqs. (1) and (2) can be written as

$$\ddot{\xi}_n + D\dot{\xi}_n + \Omega\xi_n = -P^T G\dot{\eta} \tag{4}$$